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Finite-Temperature Renormalization Group Predictions: The Critical Temperature, Exponents and Amplitude Ratios

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Abstract: $\lambda\varphi^4$ theory at finite temperature suffers from infrared divergences near the temperature at which the symmetry is restored. These divergences are handled using renormalization group methods. Flow equations which use a fiducial mass as flow parameter are well adapted to predicting the non-trivial critical exponents whose presence is reflected in these divergences. Using a fiducial temperature as flow parameter, we predict the critical temperature, at which the mass vanishes, in terms of the zero-temperature mass and coupling. We find some universal amplitude ratios which connect the broken and symmetric phases of the theory which agree well with those of the three-dimensional Ising model obtained from high- and low-temperature series expansions.

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1 Introduction

The problem of ultraviolet (UV) divergences is well known to have its solution in the proper renormalization of a theory. What is not so well appreciated is that infrared (IR) divergences can equally well be treated by such methods. In finite temperature field theory UV divergences are temperature independent while the IR divergences are temperature dependent. Thus for the satisfactory removal of these finite temperature divergences it is necessary to have a subtraction scheme which is also temperature dependent. When this subtraction scheme is associated with a multiplicative renormalization group then the flow equations of that group will of necessity be temperature dependent. This is a special case of a more general situation where IR singularities depend on "environmental" parameters. In such settings the flow equations of a well adapted renormalization group (RG) should also depend on those parameters. Such renormalization groups we call "environmentally friendly" renormalization groups.

The techniques of "environmentally friendly renormalization" [1, 2] offer a quite general approach to investigating, both qualitatively and quantitatively, the change from one type of critical behaviour to another as an "environmental" parameter is varied. Such a change of critical behaviour is known, in the terminology of critical phenomena, as a crossover. In the context of finite-temperature field theory, a temperature dependent reparametrization defined by normalization conditions provides a method of tracking the evolution of the effective degrees of freedom as a function of both scale and temperature. The resulting finite-temperature renormalization group [3] is therefore, in principle, environmentally friendly.

The problem of finite temperature field theory in its Euclidean formulation is exactly that of critical phenomena on the manifold $R^3 \times S^1$. In this setting it falls under the heading of critical phenomena with finite size corrections. The finite size of the S^1 is $L = 1/T$ (more conventionally one uses the symbol $\beta = 1/T$). In the series of papers [1, 2, 4, 5] a one-parameter family of reparametrizations at fixed temperature T, parametrized by an arbitrary fiducial finite-temperature mass, was considered. This rendered the complete crossover for all values of T accessible in one uniform perturbation expansion. The crossover was analyzed to two loops and the phase transition shown to be second order, characterized by three-dimensional critical exponents. More recently [6] we found reliable predictions, for the critical regime in terms of the zero-temperature parameters of the theory, by considering a one parameter family of renormalized couplings parametrized by an arbitrary fiducial temperature τ.

In the neighbourhood of the critical point where the symmetry is restored the mass and coupling vanish continuously at T_c and behave as $(f_1^{\pm})^{-1}|T - T_c|^{\nu}$ and $l^{\pm}|T - T_c|^{\nu(4-d_c)}$ respectively, as expected from critical phenomena (and in agreement with [1]). Here, d_c is the reduced dimension in the critical regime, ν and η are characteristic exponents, $f_1^{\pm}$ and $l^{\pm}$ are amplitudes, the $\pm$ referring to above and below the critical temperature respectively. For $\lambda\phi^4$ theory and the physical dimension $d = 4$, one

finds $d_c = 3$ and our two loop results gave $\nu = 0.639$ and $\eta = 0.0329$ [2] while the best available estimates for these exponents are $\nu = 0.6310 \pm 0.0015$ and $\eta = 0.0375 \pm 0.0025$ [15].

Amplitudes differ above and below the critical point, however certain ratios of these amplitudes, like critical exponents, are universal numbers. In the present context this means they are independent of the zero-temperature mass and coupling which we use to parametrize the theory. We found [6] that the amplitude ratio $f_1^+/f_1^- = 1.92$ which is in good agreement with the best series expansion results of Liu and Fisher [7] who obtain 1.96 ± 0.01 (this should be compared with the result 1.41 obtained from a tree level analysis where fluctuations are ignored and 1.91 from the ϵ expansion at order ϵ^2 [8] for $\epsilon = 1$). The other principal amplitude ratio we studied was that associated with the two point vertex function $\Gamma^{(2)}$ at zero momentum which we denote by M^2. Near T_c it takes the form $M_\pm^2 = C^\pm |T - T_c|^\gamma$ and we found that $C^+/C^- = 4$ while the best estimates of Liu and Fisher from series expansions give $C^+/C^- = 4.95 \pm 0.15$.

2 Finite-temperature renormalization

We consider a φ^4 theory in equilibrium with a thermal bath at temperature $T = 1/\beta$. It is described by the Euclidean action

$$S[\varphi_B] = \int_0^\beta dt \int d^{d-1}x \left[\frac{1}{2}(\nabla\varphi_B)^2 + \frac{1}{2} M_B^2 \varphi_B^2 + \frac{\lambda_B}{4!}\varphi_B^4 \right]. \tag{1}$$

The effective potential V, or effective action (Γ) per unit volume is a function of the renormalized field expectation value $\bar\varphi = Z_\varphi^{-1/2}\bar\varphi_B$ and provides a convenient summary of many of the thermodynamic properties of the model. To access an arbitrary value of $\bar\varphi$ it is necessary to couple in a constant external current J, which will produce this expectation value. In fact the effective potential is a function of $\bar\varphi^2$ and the relation between $\bar\varphi$ and J is given by the equation of state

$$\Gamma^{(1)} = \Gamma_t^{(2)}\bar\varphi = J, \tag{2}$$

which also serves to define $\Gamma_t^{(2)}$ again as a function of $\bar\varphi^2$. We further define $\Gamma_t^{(4)}$ through

$$\Gamma^{(2)} = \Gamma_t^{(2)} + \frac{\Gamma_t^{(4)}}{3}\bar\varphi^2, \tag{3}$$

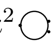
and more generally each n-point vertex function admits a decomposition in terms of vertex functions $\Gamma_t^{(n)}$ which are functions of $\bar\varphi^2$.

For $J = 0$ there are two possible solutions of (2): Either $\bar\varphi = 0$ with $\Gamma_t^{(2)} \neq 0$ or $\bar\varphi = \bar\varphi_0 \neq 0$ and $\Gamma_t^{(2)} = 0$. For fixed temperature it is natural to consider this

model and hence the effective potential as a function of both $\bar{\varphi}$ and M_B^2. In the entire $(M_B^2, \bar{\varphi})$ plane both branches of (2) are realized. If we locate our origin at the point where the model switches from one branch to another, then we can describe the plane in terms of new co-ordinates $(t_B, \bar{\varphi})$ where $t_B = M_B^2 + \Delta(T)$. It turns out that it is necessary to multiplicatively renormalize t_B in a similar fashion to $\bar{\varphi}$ by defining $t = Z_{\varphi^2}^{-1} t_B$. In fact to control the infrared divergences that necessarily arise close to the origin of the $(t_B, \bar{\varphi})$ plane, which corresponds to the location of a second order phase transition, it is necessary to have both Z_φ and Z_{φ^2} temperature dependent. If one changes T then the picture remains qualitatively the same but the origin and detailed shape of the co-existence curve change somewhat.

We define the Z factors above by the following normalization conditions

$$
\begin{aligned}
\frac{\partial}{\partial p^2} \Gamma_t^{(2)}(p, \bar{\varphi}(\tau,\kappa), t(\tau,\kappa), \lambda(\tau,\kappa), \tau)\Big|_{p=0} &= 1, && (4) \\
\frac{\Gamma^{(2)}(0, \bar{\varphi}(\tau,\kappa), t(\kappa,\tau), \lambda(\tau,\kappa), \tau)}{1 + \frac{\bar{\varphi}^2}{3} \frac{\partial}{\partial p^2} \Gamma_t^{(4)}(\tau,\kappa)\Big|_{p=0}} &= \kappa^2(\tau), && (5) \\
\Gamma_t^{(2,1)}(0, \bar{\varphi}(\tau,\kappa), t(\tau,\kappa), \lambda(\tau,\kappa), \tau) &= 1, && (6) \\
\Gamma_t^{(4)}(0, \bar{\varphi}(\tau,\kappa), t(\tau,\kappa), \lambda(\tau,\kappa), \tau) &= \lambda(\tau,\kappa). && (7)
\end{aligned}
$$

These normalization conditions allow us to present the two different schemes that we have used. The first corresponds to flowing κ while τ is set to the physical temperature T. It is convenient to define $h = 4\lambda\kappa^2$  and $z = \frac{\kappa}{T}$ in which case the flow equations become

$$
\begin{aligned}
\kappa \frac{dh}{d\kappa} &= \beta(h, z), && (8) \\
\kappa \frac{d \ln Z_{\varphi^2}^{-1}}{d\kappa} &= \gamma_{\varphi^2}(h, z), && (9) \\
\kappa \frac{d \ln Z_\varphi}{d\kappa} &= \gamma_\varphi(h, z). && (10)
\end{aligned}
$$

In [2] the functions $\beta(h,z)$, $\gamma_{\varphi^2}(h,z)$ and $\gamma_\varphi(h,z)$ were calculated to two loops. Then using Padé resummation techniques the resulting equations were solved to find quite accurate expressions for the resulting Wilson functions. These capture the dominant part of the critical behaviour over the entire range of temperatures and especially in the neighbourhood of the critical temperature. For example the relationship between the inverse correlation length and the parameter t that tells us the deviation from the critical region is given by

$$
t(m, \kappa) = \kappa^2 \int_0^m \frac{dx}{x} (2 - \gamma_\varphi) e^{\int_\kappa^x (2 - \gamma_{\varphi^2}) \frac{dy}{y}}. \qquad (11)
$$

The principal quantity that is missed in this approach is the critical temperature or equivalently the quantity $\Delta(T)$ described above.

The second prescription involves flowing τ at fixed M_B and allows us to predict the critical temperature in terms of the zero temperature parameters, precisely as one would wish to do in the case of the standard model. In this prescription we choose to follow the flow of $M^2 = \Gamma^{(2)}\Big|_{p=0}$. The differential equations which describe an infinitesimal change in normalization point with fixed bare parameters are

$$\tau\frac{d\ln Z_\varphi(\tau)}{d\tau} = \gamma_\varphi, \qquad \tau\frac{d\ln Z_{\varphi^2}^{-1}}{d\tau} = \gamma_{\varphi^2}, \qquad \tau\frac{dM^2(\tau)}{d\tau} = \beta_M, \qquad \tau\frac{d\lambda(\tau)}{d\tau} = \beta_\lambda. \tag{12}$$

For $J = 0$ the flow functions β_M, β_λ and γ_φ take different functional forms above and below the branching point which corresponds to $T = T_c$, the critical temperature of the model. The flow functions to one loop, are given by

$$\gamma_\varphi = 0, \tag{13}$$

$$\gamma_{\varphi^2} = -\frac{1}{2}\lambda\tau\frac{d}{d\tau}\bigcirc, \tag{14}$$

$$\beta_M = \begin{cases} \frac{\lambda}{2}\tau\frac{\partial\bigcirc}{\partial\tau}, & \tau > T_c \\ -\lambda\left(\tau\frac{\partial\bigcirc}{\partial\tau} + \frac{3}{2}M^2\tau\frac{\partial\bigcirc}{\partial\tau}\right), & \tau < T_c, \end{cases} \tag{15}$$

$$\beta_\lambda = -\frac{3}{2}\lambda^2\tau\frac{d}{d\tau}\bigcirc. \tag{16}$$

The symbol $\bigcirc$ with k dots stands for the one-loop diagram with k propagators, without vertex factors, and zero external momentum.

After solving the flow equations we are free to choose the reference temperature τ equal to the actual temperature T of interest. In fact this is essential if one wishes to obtain perturbatively sensible results for physical quantities [9]. Because of the renormalization conditions (5, 7) the parameters $M(T)$ and $\lambda(T)$ therefore describe the behaviour of the vertex functions $\Gamma^{(2)}$ and $\Gamma_t^{(4)}$ at zero momentum. With these equations one is also able to determine the critical temperature in terms of the zero-temperature parameters $m(0)$ and $\lambda(0)$. As may be expected on dimensional grounds T_c is proportional to $m(0)$, the constant of proportionality being a function of $\lambda(0)$. Our critical temperature is larger than that found by others [10, 11, 12] and the value $t = \sqrt{12}$ is approached in the zero-coupling limit. Our results for the critical temperature are most closely comparable with "coarse-graining" type renormalization groups such as used in [12]. They also found the critical temperature increases as a function of $\lambda(0)$ but at a rate substantially less than we find. Also their results are dependent on the particular type of cutoff function used.

If we take values for $\lambda(0)$ and $M(0)$ to be those associated with estimates for the equivalent parameters in the Higgs sector of the standard model [13] we find, for $\lambda(0) = 1.98$ and $M(0) = 200$GeV that $T_c = 613$GeV. By comparison Dolan and Jackiw's result [10] gives 492.4GeV. For $\lambda(0) = 3.00$ and $M(0) = 246$GeV the corresponding results are $T_c = 639$GeV and 492.0GeV respectively.

Since $M(T) \ll T$ near T_c the finite-temperature four-dimensional theory reduces there to a three-dimensional Landau-Ginzburg model, in accord with finite size scaling (see [14] for a review). In the neighbourhood or the critical temperature the general vertex functions have the form

$$\Gamma^{(n)}_{\pm} = \gamma^{(n)}_{\pm} |T - T_c|^{\nu\left(d_c - n\frac{d_c - 2 + \eta}{2}\right)}, \tag{17}$$

where $d_c = d - 1$ is the reduced dimension at the critical point, ν and η are critical exponents and $\gamma^{(n)}_{\pm}$ are amplitudes. The appearance of the critical exponent ν is unusual in particle physics. It reflects the need for composite operator renormalization and the physical dependence of $m(T)$ on temperature. This ensures that the exponent ν is physically accessible in finite temperature field theory whereas it is usually not experimentally observable in a particle physics context as the dependence of the renormalized mass on the bare mass is experimentally inaccessible. As we see here, however, the exponent ν in fact plays a highly significant role near the critical point.

From the solutions (16) by noting that the function $\bigcirc$ behaves like $T/8\pi M$ we see that as the critical point is approached, the coupling vanishes [1, 2, 4, 12]. Hence, near the critical temperature we have $M^2_{\pm} = (C^{\pm})^{-1}|T - T_c|^{\gamma}$ with $\lambda_{\pm} = l^{\pm}|T - T_c|^{\nu}$ and $m_{\pm} = (f_1^{\pm})^{-1}|T - T_c|^{\nu}$, where we use the notation of Liu and Fisher [7] for the amplitudes. and $\gamma = \nu(2 - \eta)$. Our temperature flow equations give the one loop values for the exponents $\nu = 1$ and $\eta = 0$, which are not as good as those obtained by flowing the mass parameter where our two loop Padé results [2] for these exponents are $\nu = 0.639$ and $\eta = 0.0329$. We expect, however, that the agreement between the two schemes should improve at higher orders. The exponents ν and η have been estimated by different methods (see [15]) with the results $\nu = 0.6310 \pm 0.0015$ and $\eta = 0.0375 \pm 0.0025$.

The temperature flow scheme gives the amplitude ratios

$$\frac{C^+}{C^-} = 4, \qquad \frac{f_1^+}{f_1^-} = 2\sqrt{\frac{12}{13}} \approx 1.92, \qquad \frac{l^+}{l^-} = \frac{1}{2}. \tag{18}$$

The fact that these are not one is indicative of a cusp in the graphs of mass and coupling versus temperature as the theory passes through the critical temperature. These ratios are universal numbers analogous to the critical exponents. The best estimates for the amplitude ratios are the high- and low-temperature series expansion results of Liu and Fisher [7] who find

$$\frac{C^+}{C^-} = 4.95 \pm 0.15, \qquad \frac{f_1^+}{f_1^-} = 1.96 \pm 0.01, \tag{19}$$

which our results are in good agreement with. By comparison: at tree level (mean field theory) $C^+/C^- = 2$ and $f_1^+/f_1^- = 1.41$, whilst in the ϵ expansion at order ϵ^2, assuming dimensional reduction, $C^+/C^- = 4.8$ and $f_1^+/f_1^- = 1.91$ [8].

We see here that the amplitude ratios are substantially better than the results for critical exponents. This indicates a complimentarity between the current approach

of flowing the environment, temperature, and that of [1, 2] where the flow parameter was the finite temperature mass. At one loop the latter group gives better results for amplitudes whereas the former gives better results for exponents, but both schemes should converge to the same results as one goes to higher orders.

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